AMERICAN MUSEUM NOVITATES

Published by
Number 1190 THE AMERICAN MUSEUM OF NATURAL HISTORY September 30, 1942
New York City

A STANDARD FREQUENCY DISTRIBUTION METHOD

BY GEORGE GAYLORD SIMPSON AND ANNE ROE

INTRODUCTION

The greater part of the data of the observational sciences and many experimental data consist of measurements and counts of variates. The proper routine treatment of these is to tabulate them as frequency distributions and to apply methods of group description, inference and comparison. Various statistical procedures have been devised and are in general use for these purposes, but the student who is primarily, say, a zoologist or a psychologist, rather than a statistician, frequently experiences difficulties in selecting suitable methods and in keeping his records in condensed, readily comparable form. There has particularly been a dearth of information on quick, easy tests that may supplement, precede or sometimes obviate the more efficient but more intricate procedures of the advanced statistician.

The conscientious worker is sometimes in the dilemma of having to use procedures perhaps unduly complex and probably imperfectly understood, or having to confine himself to unreliable and unrevealing methods. In publication there is another dilemma: publishing raw data in full uses much space and shirks the responsibility of analysis and determination of meaning, but more summary presentation may not provide the data needed by another student, may not be immediately comparable with related figures in the literature, and may not sufficiently substantiate the conclusions based on the data. Frequency distributions that are statistically similar may also appear very different, and much experience is required to avoid being misled by extraneous differences.

These and other difficulties may be, if not solved, at least ameliorated by a suitable standard system of treating frequency distributions. Perfection is not to be expected: no single system can be applicable in all cases, and the ideal of uniformity and comparability is in some measure opposed to the ideal of maximum efficiency and specific suitability. Neither the system proposed in this paper nor any other can substitute for skill and judgment, but a tool can be supplied for the use of skill and a basis for the easier attainment of judgment.

The principal aims of the present method are:

To provide a convenient standard form for recording and publishing data on frequency distributions.

To bring together concisely all the data that are likely to be needed for further uses.

To specify a limited but widely useful armament of statistics with graded choice between simple but relatively inefficient and efficient but relatively complex methods of estimation.

To provide a means of graphic presentation in which differences due to size of sample, units of measurement, mean value and other factors extraneous from the given point of view are systematized and minimized or eliminated.

To combine with this a standard normal curve that will be roughly fitted to the graphic distribution without any calculation.

To provide simple graphic and tabular means for estimating the statistical significance of deviations, and making related tests, without calculation.

To facilitate various other special uses such as the scoring of test results with little or no calculation.

These aims are furthered by the provision of a printed chart or record blank, by the explanations and methods of the following text and by the tables printed in this paper.

RECORD OF OBSERVATIONS

The original recording of measurements or similar observations has so many different requirements in different cases that no standard form is necessary or possible. On the standard chart the frequency distribution is entered as such, without entry as individual observations. Either or both of the two forms of the absolute frequency distribution may be entered: distribution against the scale of measurement and distribution in abscissal deciles of the standard range. From the latter a relative distribution by percentages of total frequency is derived and entered both numerically and graphically. The significance and easy calculation of these various forms of distribution are explained on subsequent pages.

Additional basic data always to be recorded and to be included in any publication are the total frequency of the given sample, N, and its observed range, OR (the highest and lowest observations and the difference between them).

RECORD OF SAMPLE STATISTICS

The proposed standard set of sample statistics includes standard range, SR, arithmetic mean, M, standard deviation, σ, and coefficient of variation, V. Except standard range, on which see below, these are sufficiently defined and explained in Simpson and Roe (1939). In some cases it is not necessary to include all these statistics, but it is preferable to do so unless some of them obviously have no meaning for the particular sort of data involved. They take up little space and even if not needed at the time they are likely to be necessary for later comparisons, especially in publication when the full list of original observations is not provided. In most cases the 1% points or standard errors of each of these statistics should also be entered.

Two methods of obtaining these statistics are possible: by calculation from the distribution and by estimation from the observed range. For the first method (again excepting standard range) the calculations are well known and are fully explained in Simpson and Roe (1939) and numerous other texts. This is the most efficient and reliable method, but it is relatively laborious. If the data are to be published, it is usually, but not always, advisable to use this method. The second method is explained below and more fully in Simpson (1941). This method is very quick and easy, but it is less efficient and has considerably broader limits of probability. It is particularly useful as a quick preliminary means of judging what the significance of the data is likely to be, and which variates are most promising for further consideration. In some cases its results are decisive enough to obviate the necessity of more complex calculation. If statistics are entered without specification, they are to be understood as calculated from the distribution. If they are estimated from the observed range, the symbol (OR) should be inserted.

The only other calculation that usually is required is that of the limits of the abscissal deciles, explained on a later page. Entry of the data on the standard chart automatically provides estimates of frequency quartiles, deciles and percentiles, of d/σ (deviation in units of standard deviation), of P (sampling probability) and of deviations in relative frequency from the normal distribution, all of which can be read directly from the graph and from tables without calculation.

ESTIMATION OF STATISTICS FROM OBSERVED RANGE

The crude but often useful estimation of statistics from observed range has been fully discussed and exemplified in a previous paper (Simpson, 1941) and this need not be repeated. In order to have all necessary working data in one place, the symbols and formulas for both methods of obtaining the statistical values are given:

Calculation from distribution	Estimation from observed range
$SR(SD) = 6.48\sigma$	$SR(OR) = F(OR)$
$M = \Sigma(X)/N$	$M(OR)$ = (Highest observed value) $-(OR)/2$ = (Lowest observed value) $+ (OR)/2$
$\sigma = \sqrt{\Sigma(d^2)/N}$	$\sigma(OR) = F(OR)/6.48$*
$V = 100\sigma/M$	$V(OR) = 100\sigma(OR)/M(OR)$

SYMBOLS

$SR(SD)$, standard range from standard deviation.
$SR(OR)$, standard range from observed range.
F, factor from Table I (of this paper) corresponding with given sample size.
OR, observed range.
M, arithmetic mean (calculated).
$M(OR)$, arithmetic mean from observed range.
Σ, summation of all terms designated in following parenthesis.
X, any one value of the variate.
N, total frequency of given sample.
σ, standard deviation (calculated).
$\sigma(OR)$, standard deviation from observed range.
d, deviation from mean ($= X - M$).
V, coefficient of variation (calculated).
$V(OR)$, coefficient of variation from observed range.

STANDARD RANGE

The concepts and methods of standard range have also been fully discussed and exemplified in Simpson (1941). All that is necessary here is a succinct statement for the purposes of practical use of the proposed standard system. The more detailed paper should be read for fuller comprehension, and the present summary will suffice for subsequent reference.

The theoretical infinite normal distribution is unlimited and so has infinite range, but real populations and samples of them are finite in frequency and in range. Range of finite groups is variable but has a constant central tendency or mean value. This constant depends on the dispersion of the population (as best measured by the standard deviation) and on the size of the sample or real population. Since the range of samples from the same population has quite different mean values for samples of different sizes and has no constant relationship to the population range, the observed or sample range has no meaning for population studies unless related to sample frequency and cannot be compared for different samples unless reduced to some one standard frequency.

For various practical reasons, the standard frequency chosen is 1,000. The standard range is an estimate, derived from a given sample, of the mean value of the range in samples or finite populations of 1,000. Two principal methods of estimation are provided, as listed in the preceding section. The mean observed range in samples of any given size has a constant ratio to the mean range in samples of the standard size. These ratios are tabled in Table I of this paper. (For estimation of sampling probabilities see further data in Simpson, 1941.) Observed range multiplied by the factor in the table thus gives a rough estimate of standard range, $SR(OR)$. For any given sample size, the mean range divided by population standard deviation is constant. For the standard frequency 1,000, this ratio is 6.48. Therefore, if the population standard deviation has been estimated from the sample, multiplying this by 6.48 gives an estimate of standard range, $SR(SD)$. $SR(OR)$ and $SR(SD)$ both are consistent estimates of the same constant, and they tend always to have the same value, but $SR(OR)$ is much the more variable of the two and is, in this sense, less reliable.

(For the statistical theory involved, derivations of formulae, and graphs and tables in different form from those used here, see Tippett, 1925, and E. Pearson, 1932.)

* A somewhat more direct estimate, arithmetically equivalent to this, is possible by use of Table I in Simpson, 1941, not reprinted here, based on the more extensive table in Tippett, 1925.

ABSCISSAL DECILE DISTRIBUTIONS

Sample distributions represent population distributions in an imperfect but systematically variable way, and statistics derived from samples are used as estimates of the corresponding characters of the populations as a whole. Judgment as to these sample-population relationships and their use for estimates of probability of deviations, and the like, are customarily derived from various statistical tables or by diverse graphic methods that approximate the observed data to idealized normal (or sometimes other) curve data. Graphic approximation has many advantages in conciseness, ease of visualization and frequently in ease or elimination of much calculation.

The basic principle of such graphic methods is to divide the observed frequency distribution into a convenient number of defined classes and to compare the results with approximately fitted ideal distributions and statistics. The classes are decile classes if 10 in number and percentile classes if 100 in number. They may be classes of equal frequency, and then are frequency decile and percentile classes or ordinate decile and percentile classes,[1] that

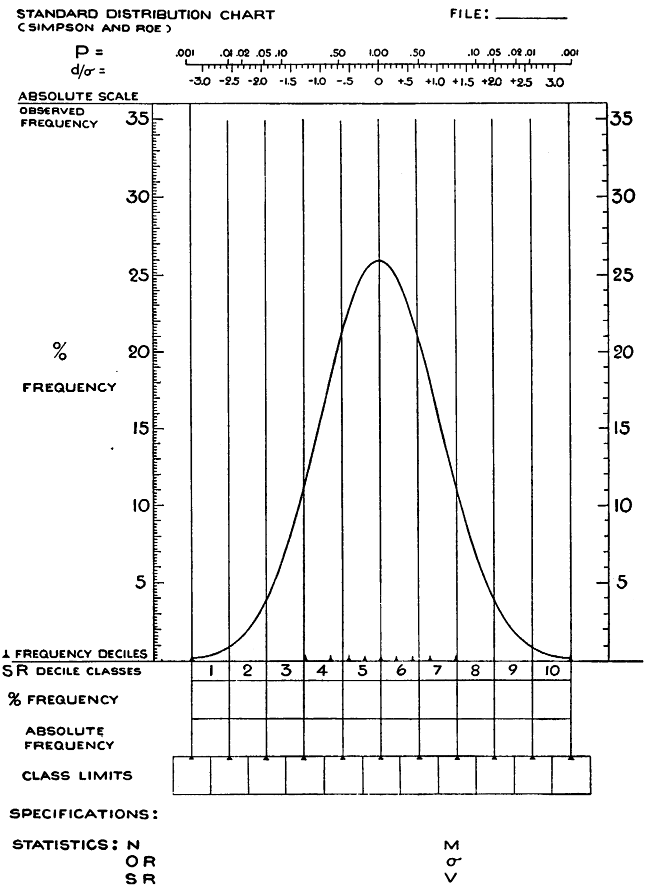

Fig. 1. Standard chart. As reproduced for work-sheets and filing, the chart is printed on loose leaves 8 by 11 inches and each abscissal (or *SR*) decile class is 1 cm. in width.

[1] In the literature unqualified deciles (and percentiles) are usually assumed to be frequency deciles, but the usage is ambiguous.

is, a frequency decile class contains one-tenth of the total frequency of the distribution. The variable to be treated and compared is then the span or range of values covered by each decile class (or percentile class, hereafter understood). Simple graphic use of ordinate deciles necessitates plotting of the cumulative distribution, which is disadvantageous because such a plot does not easily convey the real form and character of the distribution.

Instead of having equal frequency, the classes may be defined as having equal range of values of the variate, that is, they are equal subdivisions of the abscissa of the usual graph of a frequency distribution, and deciles so defined are abscissal deciles. Then the variable for representation, study and comparison is the frequency included in each abscissal decile class. This is the method used in this paper.

Full specification requires statement not only that abscissal deciles are used but also whether the frequencies are absolute or relative and how the span to be divided into ten equal parts is determined. Use of absolute frequencies, those observed in the given sample, is of course a customary ordinary method of representation of a distribution, for instance, in a histogram, but it will not serve the purposes here in view because its use means that distributions with identical statistics may appear very different and because every sample will probably approximate a different form of the normal distribution, requiring separate tabling or calculation. These disadvantages are minimized if relative frequencies are used in place of absolute. Percentages of total sample frequency will be used as a convenient and generally understood form of relative frequency.

Ordinate deciles are parameters, or constants, of the normal curve because the curve has a determinate, finite area (proportional to frequency), and this can be divided into ten equal parts. Abscissal deciles are not necessarily parameters of that curve (or of other ideal distributions), because the range of the curve is theoretically infinite and cannot be divided into finite parts. Therefore the same artifice must be employed as for the standardization of range, the curve must be taken as terminating at some specified point and as having finite range. Divisions of the observed range into ten equal parts, giving what may be called observed range or *OR* deciles, is unsuitable because the systematic variation of *OR* with *N* defeats the purpose of comparing samples of different sizes and involves other decided disadvantages.[1]

From the background of standard range, *SR*, methods, it is clear that the *SR* deciles will fulfill the requirements. *SR* is a constant of the normal distribution (does not systematically vary with *N* or other extraneous sample characters), it can easily be estimated more or less closely from samples of any size greater than two, and it almost always includes all the observed frequencies. *SR* is, then, the total abscissal distance to be used for our standard distributions. The length of each *SR* decile is $SR/10$. The lower limit of the distribution is $M - SR/2$ and the upper limit is $M + SR/2$, in which M is an estimate of the population mean from the sample. The sample mean thus falls in the middle of the diagram for all samples and coincides with the mean of the comparative normal curve. Successive decile limits are obtained by successive addition of $SR/10$

[1] The principal previously proposed abscissal decile method, that of Pearl (1940), uses modified *OR* deciles. Because it is desirable that the extreme observations be within rather than at the limits of the terminal decile classes, Pearl set the range lower limit at $OR/20$ below the lowest observation and the upper at $OR/20$ above the highest. This arbitrarily eliminates a slight defect of the method but does not eliminate the grave defect of systematic dependence on *N*. The abscissal range of Pearl's standard distribution is $OR + 2\ (OR)/20 = 1.1\ (OR)$. The decile class interval is thus $.11\ (OR)$. By this method, two samples perfectly representative of the same population may appear very different. Perfectly normal samples of any reasonable size always appear platykurtic, and very large normal samples appear leptokurtic. The standard normal curve of this method really tends to fit the sample histograms only when $N = 928$. Thus the aims of comparability and approximate curve fitting are not achieved. Moreover the sample mean does not fall in the middle of the diagram, it does not coincide with the curve mean, and its position is not determinable by inspection. Skew tends to be exaggerated, and its apparent direction may be confusing.

These imperfections are minimized or eliminated by the use of *SR* deciles in place of Pearl's 1.1 *OR* deciles. We hope and believe that our method is an improvement on Pearl's, but it grew out of study and use of the latter. It certainly implies no shortcoming in Pearl and no destructive criticism of his method, but only such further development and refinement as is always to be expected. Unfortunately Pearl's untimely death immediately after publishing his method prevented his being the one to carry it forward.

starting at $M - SR/2$, or successive subtraction of $SR/10$ starting at $M + SR/2$.

With the *SR* decile class limits determined in this simple way, the observed frequencies are entered in the classes and are converted into relative (%) frequencies by dividing by *N* (sample frequency) and multiplying by 100. This, the relative *SR* decile distribution, is the standard distribution of the present method. It may now be graphically portrayed as a histogram and compared with a standard normal curve, appropriate (with certain limitations discussed below) for any roughly normal observed distribution treated in this way and drawn once and for all on the standard chart described and illustrated on subsequent pages.[1]

The frequency deciles for the standard normal curve can also be tabulated, as in Tables II and IV (of this paper) and inserted on the chart, as in Fig. 1. These completely normal frequency deciles will not correspond exactly with those calculated from the sample but will approximate the latter for samples of fair to large size. More important, these charted frequency deciles, obtained without any calculation specifically for this purpose, are better estimates of population frequency deciles (if the population is normal or nearly so) than can be obtained by relatively laborious calculation directly from the observed values.

At the top of the chart there is a scale of deviations in terms of standard deviations, d/σ, and of corresponding sampling probabilities, *P*. *P* gives as a fraction of 1.00 the probability of random drawing from the normal population of an observation with equal or greater deviation from the mean. For closer estimation than by inspection of the chart, these values are also given in Tables II, III and V. The d/σ and *P* values are those of the standard normal curve. If *SR* has been derived from standard deviation, giving *SR(SD)*, the d/σ and *P* of the graph and tables will be identical with estimated values calculated from the sample. If *SR(OR)* has been used, the graphed and tabled d/σ and *P* values will not be exactly equal to those calculated in the ordinary way because they will have been obtained from a different and less efficient (but much quicker) method of estimating population parameters. The charted values are usually sufficiently good estimates for samples of moderate or large size, $N = 15$ or more. When *N* is less than 15, it is usually preferable to obtain *P* from the distribution of *t*, see Simpson and Roe (1939, pp. 205–209).

APPROXIMATION OF HISTOGRAM AND CURVE

No real sample is ever completely normal in distribution. Judgment as to the expected agreement and disagreement of a sample distribution in hand and a normal population distribution inferred from it must depend in large part on experience, and measurement of the significance of sample deviations from expected normal distribution statistics demands rather elaborate calculation. Therefore, exact agreement of histogram and curve in the graphs of the present method not only is not to be expected but also would be suspicious if it did occur.

The curve is drawn on the hypothesis that the statistics of the sample are exactly equal to the parameters of the corresponding population. This hypothesis underlies any use of normal distribution statistics, and the curve as drawn shows exactly what the implications of this hypothesis are in any given case. Thus the curve is useful even when it does not fit the histogram closely. (Even aside from random sampling fluctuations, the plotted curve is not necessarily the exact normal curve best fitting the data, but it is very nearly so—

[1] The similar chart prepared by Pearl and illustrated in his book (Pearl, 1940, Fig. 109) cannot be used for the *SR* decile distribution because a distinctly different curve results from his different method of limiting the abscissal decile classes. His range has the limits at $M \pm 3.54\sigma$ on the postulate that 1.1 *OR* tends to be, or averages, 7.08σ. This is true only when $N = 928$. Our range has limits at $M \pm 3.24\sigma$ and assumes that *SR* tends to be 6.48σ, which is always true of samples from a normal population regardless of the value of *N* (unless, of course, *N* is 1 or 2).

close enough for any but most elaborate studies of very large samples.)

Exact equivalence of histogram and curve would result in the curve's intersecting the top of each rectangle of the histogram near its midpoint. The method of curve fitting that has been used does not place this intersection exactly in the middle, but the difference is very slight in comparison with random sampling fluctuations, even for very large samples, and need not be taken into account. The probable significance of deviations from this theoretical fit must, of course, be judged in the light of sample size, and there is no way of determining such significance directly from a graph of this type. In general, purely random, non-significant deviations of histogram from curve will be greater with small samples than with large. With a little experience, inspection will suggest whether significance is likely, and this may then be tested by special calculation if pertinent for the problem.

If the two middle histogram classes fall distinctly below the curve, the distribution represented by the sample is likely to be platykurtic, and if they rise distinctly above the curve (but observed frequencies do occur in most classes), it is likely to be leptokurtic. If the histogram is approximately symmetrical, with the two middle classes about equal, no skew is shown. The mode is in the decile class represented by the highest rectangle of the histogram, and with no skew it falls between the middle classes, coinciding with the mean which is fixed at this point. If any one decile rectangle is decidedly higher than all the others, the population represented by the sample is likely to be skewed. If the mean, the midline of the graph, is to the right of this modal decile class, the skew is to the right, or positive. If the mean is to the left of the mode, the skew is to the left, or negative. Very strong skewness of course invalidates the use of normal curve statistics, and the graph itself will suggest whether these may properly be used. If the modal decile class is one of the two middle classes, class 5 or 6, normal curve statistics are usually sufficiently good approximations, and they may still be used if the modal decile class is the second from the mean, class 4 or 7 of the graph, provided that the asymmetry is not very pronounced and especially if the sample is rather small. If the modal decile class is strongly defined and is class 4 or 7 of the graph, use of normal curve statistics is questionable at least, and if a modal decile class can definitely be recognized and is class 1, 2, 3, 8, 9 or 10, those statistics should not, as a rule, be used.

Very large samples may occasionally include observations outside the standard range and therefore outside the *SR* deciles. This is very unlikely unless N is in the hundreds, which is seldom the case, but it can occasionally happen even with samples of moderate size. On an average, if the populations are approximately normal, only one observation in a thousand will be outside the *SR* deciles. In practice when such external frequency does occur, it may be thrown in "over" and "under" classes beyond the first and tenth *SR* decile classes. Space is left for these on the standard chart. Such occurrences do not invalidate this method or any comparisons based on it. On the contrary, the characteristics of the normal distribution require that such cases should arise occasionally if comparison in standard form is to be valid.

On the other hand, with samples of moderate or small size it will happen more often than not that the terminal *SR* decile classes, classes 1 and 10, will have zero frequency, and for very small samples, N less than 10, this will almost always be true. This, too, is a necessary consequence of correct and valid representation of the relationship between the given sample and the hypothetical population.[1] The whole frequency of a sample always lies within a smaller range than the whole frequency of the population from which it is drawn, and the occurrence of empty terminal classes merely recognizes and illustrates

[1] *OR* decile methods never have external frequencies and always have positive frequencies in the two terminal classes. This is a valid way to represent the character of the sample, as such, but it results in incorrect and misleading representation of the relationship of sample to population. It permits comparison of the samples themselves, which is seldom really desired, but it leads to false conclusions as to the comparison of populations, which is almost always the real aim.

this fact which is fundamental in drawing sound inferences. This peculiarity may, however, result on occasion and with unduly small samples in radical discrepancy between the sample histogram and the population curve. For instance, if all observations happened to fall within one *SR* decile of the mean, which is never likely with more than two observations but can happen with as many as eight, the two middle decile classes would have 100% of the sample frequency, but the corresponding curve area is only 48%. The example is extreme, and such discrepancies are very unusual.

An advantage of the *SR* decile method is that the relationship of classes to population frequency (for a normal population) is constant,[1] so that a definite criterion of possible approximation is available. If *OR* overlaps only the middle four *SR* decile classes, 100% sample frequency represents 81% population frequency, so great a discrepancy that the graphic approximation cannot be good. If six *SR* decile classes have observed frequencies, 100% sample frequency represents 95% population frequency, which is close enough for adequate approximation. For instance, the following relatively good agreement is possible:

SR decile classes	1	2	3	4	5	6	7	8	9	10
Observed frequencies, %	0	0	8	17	25	25	17	8	0	0
Normal frequencies, %	1/2	2	7	16	24	24	16	7	2	1/2

With observations in eight or more classes the agreement in integral percentages can be complete.

The number of classes likely to have observed frequencies is a function of sample size, *N*. For any value of *N*, the number of classes with observed frequencies will, of course, vary in different samples, but it will tend toward a constant average and 1% points (or fiducial limits) can be calculated. The approximate relationships are as follows:

N	Mean number of *SR* decile classes overlapped by *OR*	Numbers likely to be exceeded less than about once in a hundred samples: Minimum	Maximum
5	4	2	8
6	4	2	8
7	6	2	8
8	6	2	8
9	6	3	8
10	6	3	8
15	6	3	10
20	6	4	10
25	8	4	10
30	8	5	10

Thus even samples with $N = 30$ do not give complete assurance that six *SR* decile classes will have observed frequencies, but this is more likely than not with $N = 7$ or more. In practice, all ten *SR* decile classes will not have observed frequencies with *N* smaller than about 15.

Another limitation on the sample histogram as an approximation of the normal curve is inherent in the fact that frequency counts are integral and that this discontinuity limits the scale of possible percentage values. Thus if $N = 5$, there can be no relative frequency value between 0 and 20%, but in a normal population the relative frequencies are between 0 and 20% in eight of the ten *SR* decile classes. Samples from a normal population with $N = 10$ and with $N = 14$ could be distributed in six *SR* decile classes as follows:

SR decile classes		1	2	3	4	5	6	7	8	9	10
Observed frequencies	$N = 10$	0	0	1	2	2	2	2	1	0	0
	$N = 14$	0	0	1	2	4	4	2	1	0	0
Same in %	$N = 10$	0	0	10	20	20	20	20	10	0	0
	$N = 14$	0	0	7	14 1/2	28 1/2	28 1/2	14 1/2	7	0	0
Normal frequencies, %		1/2	2	7	16	24	24	16	7	2	1/2

For $N = 10$, the approximation is not good enough to give a clear picture of the real form of the distribution. Although manifestly imperfect, the sample with $N = 14$ does give a fair approximation.

From such considerations, it is a useful empirical rule that at least 15 observations

[1] This is not true of *OR* deciles.

distributed in 6 *SR* decile classes or at least 20 observations in 8 or more classes are necessary for the sample histogram to give a reasonably good representation of the form of population distribution. (Of course it does not follow that the representation will be good if these minimum conditions are met.)

It is at least as important to know these limitations as to know the advantages of the *SR* decile method, but it cannot be too strongly emphasized, first, that these are limitations inherent in the sampling of populations and not peculiar to this method, and, second, that they apply only to the approximation of sample to population distributions. No matter how small or how aberrant the sample may be, the histogram is still an accurate visualization of it, the entered sample statistics are still correct, and the normal curve and its plotted statistics still provide with minimum effort the best[1] hypothesis of normal population distribution that can be based on that sample.

[1] Or as near the best as is usually necessary.

STANDARD CHART AND EXAMPLE OF ITS USE

For purposes of record, comparison and publication a standard chart has been devised and is shown in Fig. 1. The original chart is on note sheets of the standard 8 by 11 inch size, on paper that will take either pencil or ink. To facilitate interpolation, percentile scoring, etc., the standard range of the chart is 10 cm. in length and the *SR* decile interval is 10 mm. For reproduction in publications, as in the figures of this paper, reduction to 1/2, more or less, is suitable. Data not needed in publication, such as the right-hand percentage frequency scale, can be cut off or blocked out before engraving, as in Figs. 4–5.

To illustrate the use of the chart and the whole system here proposed, a concrete example will be given in detail. For purposes of illustration all alternatives are given, and no data are omitted. In actual practice only one alternative is necessary, and some data may be superfluous.

The full procedure is as follows:

1. Enter specifications. E.g.: *Peromyscus maniculatus bairdii.* Both sexes. 1 year class. Near Alexander, Iowa. Tail length in mm. (Dice's data, 1932.)

2. Calculate and enter such sample statistics as are desired, except standard range. (The calculation, itself, is not done on the chart where only its results appear.) For the example:

(a) Statistics calculated from observed distribution (computation not here shown, given in Simpson and Roe, 1939):

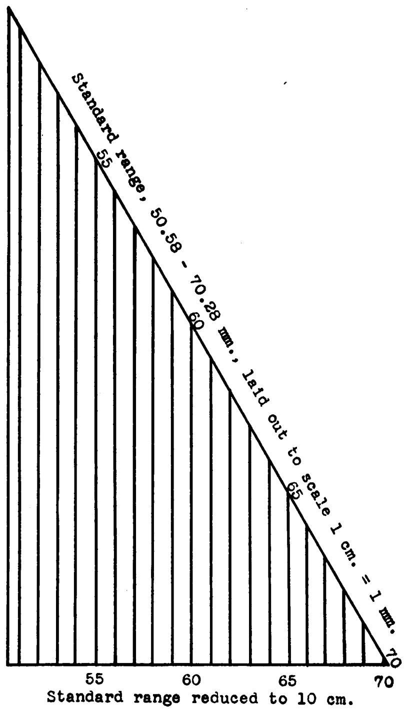

Fig. 2. Graphic method for reducing the original scale of measurement to the length *SR* = 10 cm. for entry on the standard chart. In practice, the diagram may be made on graph paper, obviating or facilitating the drawing of the parallel vertical lines. The scale is that shown on the chart in Fig. 3.

N = 86
OR = 68–52 = 16 mm.
M = 60.43 ± .33 mm.
σ = 3.04 ± .23 mm.
V = 5.0 ± .4

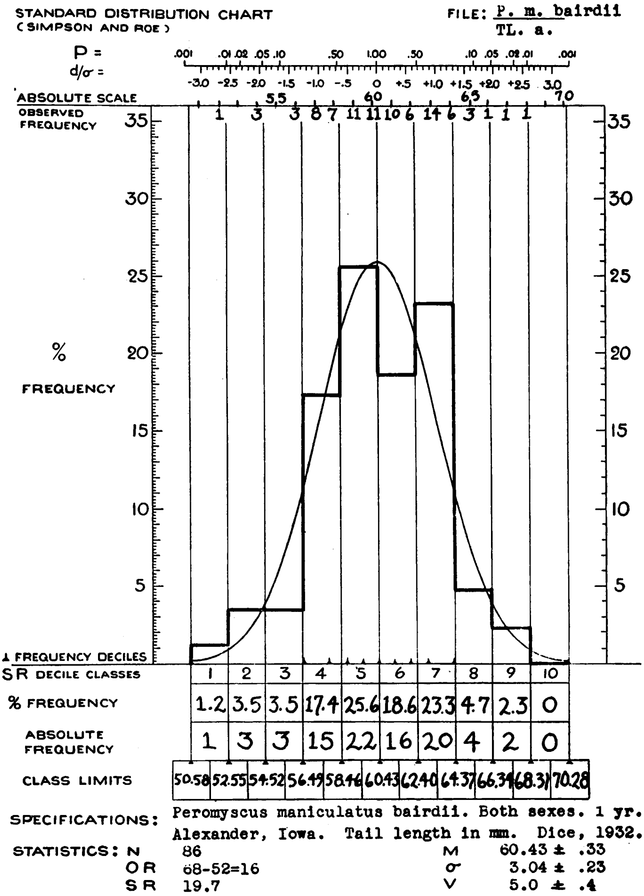

Fig. 3. A standard chart completely filled out with data on a typical zoological frequency distribution. The data are those used as an example in the text, treated in the first alternative way (a).

(b) Estimated from OR (by formulas on a preceding page of this paper):

N = 86
OR = 68 - 52 = 16 mm.
$M(OR)$ = 52 + 16/2 = 60 mm.
$\sigma(OR)$ = (1.32 × 16)/6.48 = 3.2 mm.
$V(OR)$ = (100 × 3.2)/60 = 5.3

3. Calculate and enter standard range by either of the two formulas previously given. Example:

(a) From standard deviation:
$SR(SD) = 3.04 \times 6.48 = 19.7$ mm.
(b) From observed range:
$SR(OR) = 16 \times 1.32 = 21.1$ mm.

4. Calculate and enter the lower limit of the SR decile distribution, which is $M - SR/2$. Example:

(a) $M - SR(SD)/2 = 60.43 - 9.85 = 50.58$ or,
(b) $M(OR) - SR(OR)/2 = 60.00 - 10.55 = 49.45$*

5. Calculate and enter the other limits of the SR decile classes. Example:

(a) Class interval = $SR(SD)/10 = 1.97$
$50.58 + 1.97 = 52.55$
$52.55 + 1.97 = 54.52$, etc.

* In this more roughly approximate method the decimal places are not significant, but they may be carried at this stage of the calculation. Their publication would be unwarranted unless accompanied by statement of non-significance.

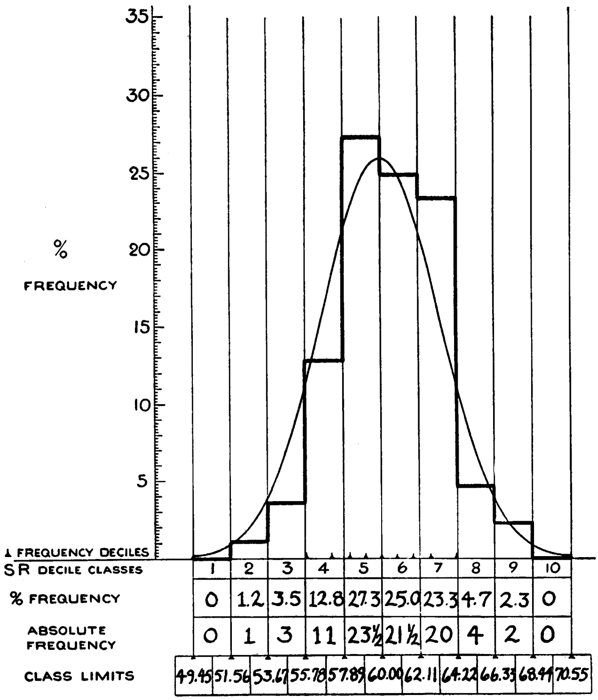

Fig. 4. Chart based on same sample as Fig. 3, treated in the second alternative way (b), as given in the text. Parts of the chart necessary for filing and record but usually unnecessary in a published illustration have been cut off.

(b) Class interval = $SR(OR)/10$ = 2.11

$$49.45 + 2.11 = 51.56$$
$$51.56 + 2.11 = 53.67, \text{ etc.}$$

6. Reduce raw measurement scale to standard length (10 cm.) and enter on chart. There are several graphic methods of doing this without computation, of which that shown in Fig. 2 for this example is perhaps the simplest.

7. Enter observed frequencies against raw measurement scale. Example, as in Fig. 3. (Steps 6 and 7 are not necessary for the system and may be omitted, but are often useful.)

8. Distribute the observed frequencies in the *SR* decile classes and enter on the chart. This may be done in two ways. The simpler is to place each observation in the *SR* decile class, the midpoint of which is nearest the observed value. If steps 6 and 7, above, have been made, this becomes simply a matter of adding the frequencies thus placed in each decile class on the chart. This easy method is generally adequate, but if the classes of measurement are relatively few and radically overlap the *SR* decile classes, the frequencies may be divided into proportional parts. Example: Figs. 3, 4.

9. Convert the absolute frequencies thus obtained into percentages of total frequency and enter on the chart. Example: Figs. 3, 4.

10. Construct the histogram on the chart. This is done directly from the entered percentage frequencies. Example: Figs. 3, 4.

This completes the procedure. Throughout, either (a) or (b) should be followed and not both, except that in some cases after using (b) for preliminary survey, it may be advisable to proceed with (a) on the same data (on another sheet); (a) is the more reliable but more laborious method.

The approximation of the histogram to the normal curve is immediately evident.

SR SCORING

This method is applicable to psychological and other test scores, as well as to zoological measurements, and can be of great value since it is possible to assign a score to any position within the standard frequency distribution, which will always mean the same position with regard to the total group, no matter what the test in question. This system offers a more exact equivalence of scoring from test to the advantage of more immediately locating the position, since it is not necessary to keep in mind the value of sigma or the relationship of sigma deviation to range. If a simple sigma score should be needed, as for comparison with data already published in that form, it is possible to convert any *SR* score into a sigma score from the chart or from the following tables, without calculation.

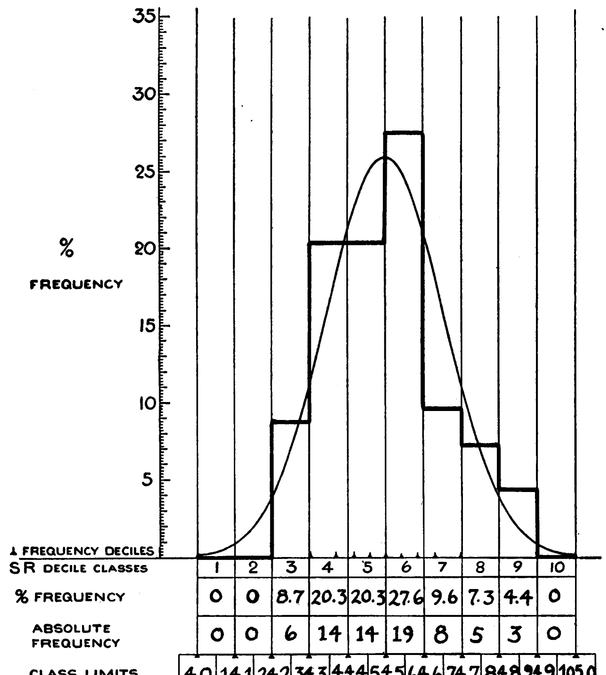

Fig. 5. Standard distribution for data by Weisenburg, Roe and McBride (1936) on Stanford-Binet vocabulary test on English-speaking, normal, white adults under 60 years of age. (See discussion of scoring and distribution in text.)

test than any proposed hitherto, and, particularly in the case of adults, a more desirable and intelligible score than an I.Q. It eliminates the effect of the size of the standardization sample on the range, since all calculations are made for a standard sample size. As compared with scoring by simple frequency deciles, *SR* scoring has all of the advantages of the more usual forms of sigma scores, and, in comparison with the other forms of sigma scores, it has

The chart, as shown, provides for ten classes. For some psychological tests, this is about as fine a discrimination as is possible, e.g., repetition of digits as usually scored. For most tests, however, a finer discrimination is not only possible but also desirable, particularly if the material is to be used in correlational studies. It is evident that doubling the number of classes is a very simple procedure, in which case there will be twenty vigintiles[1] instead of ten deciles. Unless the sample is very large, it is usually preferable to plot the histogram in 10 classes, even though 20 are used in scoring, because sampling fluctuations give less distortion in this form.

The following table and Fig. 5 show the method, as applied to Stanford-Binet Vocabulary Score (number of words passed on both lists) for the group of adult whites reported by Weisenburg, Roe and McBride (1936). There are two ways in which *SR* scores can be assigned, in place of the original raw test scores.

STANFORD-BINET VOCABULARY SCORES. NORMAL, WHITE ADULTS

In 20 classes (Vigintiles)
Class interval = 5.05

Vigintile class	Lower limit of class	Frequency
20	99.95	0
19	94.90	0
18	89.85	1
17	84.80	2
16	79.75	2
15	74.70	3
14	69.65	2
13	64.60	6
12	59.55	7
11	54.50	12
10	49.45	11
9	44.40	3
8	39.35	9
7	34.30	5
6	29.25	5
5	24.20	1
4	19.15	0
3	14.10	0
2	9.05	0
1	4.00	0

In 10 classes (Deciles)
Class interval = 10.1

Decile class	Lower limit of class	Frequency
10	94.90	0
9	84.80	3
8	74.70	5
7	64.60	8
6	54 50	19
5	44.40	14
4	34.30	14
3	24.20	6
2	14.10	0
1	4.00	0

a. The number designating the serial order of each class may be considered the *SR* score for all persons falling in that class. This gives a discontinuous series of 10 (or 20) possible scores, and these can then be averaged or treated as any others. This has the advantage of great simplicity in computing the scores. Further, if some tests must be recorded in deciles, those scored in vigintiles can readily be restated in decile terms for comparison. Following this procedure, a person who defined 58 words would be given an *SR* vigintile score of 11, or an *SR* decile score of 6.

b. If there is likely to be any necessity for finer discrimination than that afforded by vigintiles, the lower limit of the first class can be taken as the zero point of a scale, the upper limit of the 10th class as 100, the highest point of such a scale, and the scale can then be read to any degree of fineness desired. This gives a continuous scale, points on which (the *SR* scores) are directly comparable from one test to another, even though the fineness of discrimination is not the same for each test. This procedure would assign to the person correctly defining 58 words, an *SR* score of 54, or, if greater exactness is needed, 53.535. . . .

[1] In the absence of a generally adopted term and to avoid circumlocution, we shall use vigintile (from Latin *viginti*, twenty) in this obvious sense.

Standard range scoring for psychometrics and the comparative merits of these alternative *SR* scoring systems, together with specific recommendations for their choice and use, will be discussed in a separate paper. The procedure has been exemplified here for completeness in presentation of the basic *SR* data.

TABLES

The following set of tables permits a wide variety of statistical tests and transformations, saves much calculation and eliminates the need of other tables for most of the usual statistical procedures based on parameters of the normal curve.

The tabled values are those for the normal curve. All deviations, throughout the tables, are given from the mean. The decile classes are, however, numbered serially from left to right. Thus the line between decile classes 4 and 5 has deviation -1 decile from the mean, that between 8 and 9 has decile deviation $+3$, etc. Since the normal distribution is symmetrical, values need to be carried only through five deciles and may be read for either plus or minus deviations.

The tables were all calculated by the authors for this paper. In some cases, data for calculation were obtained from Pearson's "Tables for Statisticians and Biometricians."

Table I is for direct estimation of standard range and of standard deviation from observed range. $SR(OR) = F(OR)$ and $\sigma(OR) = F(OR)/6.48$, using as F the tabled value of (standard range)/(mean sample range) for the size of the given sample.

Table II contains the most basic data for interpreting *SR* decile distributions. *SR* decile deviations may be read to integers by inspection and to one decimal place by a millimeter ruler on the standard distribution sheets, where 1 mm. = $0.1SR$ decile deviation. These deviations can be converted to d/σ by the table, and sampling probability, P, can be obtained either for d/σ or for *SR* decile deviations. Tabled values of P are for $\pm$ deviations. For $+$ or $-$ deviation separately, P has one-half the tabled value. The ordinates in percentage of total frequency facilitate drawing the standard normal curve and rough judgment of goodness of fit. The table permits direct conversion of *SR* to frequency deciles and percentiles on the postulate of normal population distribution with parameters equal to the sample statistics. Deciles are tabled. To convert either *SR* or frequency class deviations to percentiles, multiply by ten. Ordinates to be read against percentiles must be divided by ten. To read percentile values on the graph, its percentage frequency scale must also be divided by ten. The d/σ and P values are the same for corresponding deciles and percentiles.

Table III is for conversion of d/σ to *SR* decile deviations.

Table IV is for the conversion of frequency deciles or percentiles to *SR* deciles, on the hypothesis of normal population distribution with parameters well estimated by the given sample statistics. *SR* decile deviations thus obtained will not be exactly equal to those charted for the same sample treated directly as an *SR* decile distribution, but if the sample is adequate and the population is, in fact, nearly normal in distribution, the differences will be small and random. Percentiles may be used by multiplying decile values by 10. For quartiles, divide by 2.5 (this may also be done in Table I). For vigintiles, multiply by 2. Other divisions of the distribution into equal classes either as fractions of *SR* or of total frequency may similarly be used by multiplying or dividing by appropriate factors, easy to determine in any case.

Table V is used to obtain the equivalent of a value of P in *SR* decile deviations and to plot values of P on the standard distribution.

Table VI permits construction of normal or population histograms not only with 10 classes, as in the proposed standard system, but also with 4, 5 or 20 classes. The table also facilitates comparison of sample histograms with the normal distribution. 0.06% of the total population frequency (or of the area of the normal curve) lies outside the standard range and therefore is not included in the classes of this method.

The following relationships permit some other conversions and special uses of the tables:

$SR(SD) = 6.48287\sigma$.
SR decile deviation $= 1.54d/\sigma$.
$P = 1 - .2$ (frequency decile deviation).

Percentage of total frequency with less than a given $\pm$ deviation from the mean $= 100 - 100P = 20$ (frequency decile deviation). Treat as positive.

Percentage of total frequency with more than a given $\pm$ deviation from the mean $= 100P = 100 - 20$ (frequency decile deviation). Treat deviation as positive.

Percentage of total frequency below a given point $= 50 + 10$ (frequency decile deviation). Retain sign of deviation.

Percentage of total frequency above a given point $= 50 - 10$ (frequency decile deviation). Retain sign of deviation.

The last two equations permit easy calculation of the percentage of frequency between or outside of any two given points in the distribution.

On the printed distribution sheets, before reduction for publication in a periodical:

$SR = 100$ mm.
1 SR decile deviation $= 10$ mm.
1 SR vigintile deviation $= 5$ mm.
1 SR percentile deviation $= 1$ mm.
$\sigma = 15.4$ mm.

TABLE I

FACTORS FOR CONVERTING OBSERVED TO STANDARD RANGE

Size of sample	Standard Range: Mean sample range
5	2.79
6	2.56
7	2.40
8	2.28
9	2.18
10	2.11
11	2.04
12	1.99
13	1.94
14	1.90
15	1.87
16	1.84
17	1.81
18	1.78
19	1.76
20	1.74
21	1.72
22	1.70
23	1.68
24	1.66
25	1.65
26	1.63
27	1.62
28	1.61
29	1.60
30	1.59

Size of sample	Standard Range: Mean sample range
31	1.58
32	1.57
33	1.56
34	1.55
35	1.54
36	1.53
37	1.52
38	1.52
39	1.51
40	1.50
42	1.49
44	1.47
46	1.46
48	1.45
50	1.44
52	1.43
54	1.42
56	1.41
58	1.41
60	1.40
62	1.39
64	1.38
66	1.38
68	1.37
70	1.36
75	1.35
80	1.34
85	1.32
90	1.31
95	1.30
100	1.29

Size of sample	Standard Range: Mean sample range
110	1.28
120	1.26
130	1.25
140	1.24
150	1.22
175	1.20
200	1.18
225	1.16
250	1.15
275	1.14
300	1.13
325	1.12
350	1.11
375	1.10
400	1.09
450	1.08
500	1.07
550	1.06
600	1.05
650	1.04
700	1.03
800	1.02
900	1.01
1,000	1.00

TABLE II

SR DECILE DEVIATIONS, DEVIATIONS IN TERMS OF STANDARD DEVIATION, PERCENTAGE FREQUENCY ORDINATES, SAMPLING PROBABILITY AND FREQUENCY DECILE DEVIATIONS

SR decile deviation	Deviation in terms of d/σ	Ordinate (% frequency)	*P*	Frequency decile deviation
0	0	25.9	1.00	0
.1	.06	25.8	.95	.26
.2	.13	25.6	.90	.52
.3	.19	25.4	.85	.77
.4	.26	25.0	.80	1.02
.5	.32	24.5	.75	1.27
.6	.39	24.0	.70	1.51
.7	.45	23.3	.65	1.75
.8	.52	22.6	.60	1.98
.9	.58	21.8	.56	2.20
1.0	.65	20.9	.52	2.42
1.1	.71	20.1	.48	2.62
1.2	.78	19.2	.44	2.82
1.3	.84	18.2	.40	3.00
1.4	.91	17.1	.36	3.18
1.5	.97	16.2	.33	3.35
1.6	1.04	15.1	.30	3.50
1.7	1.10	14.1	.27	3.65
1.8	1.17	13.1	.24	3.78
1.9	1.23	12.1	.22	3.91
2.0	1.30	11.2	.19	4.03
2.1	1.36	10.3	.17	4.13
2.2	1.43	9.4	.15	4.23
2.3	1.49	8.5	.14	4.32
2.4	1.56	7.7	.12	4.40
2.5	1.62	7.0	.10	4.47
2.6	1.69	6.3	.09	4.54
2.7	1.75	5.6	.08	4.60
2.8	1.82	5.0	.07	4.65
2.9	1.88	4.4	.06	4.70
3.0	1.94	3.9	.05	4.74
3.1	2.01	3.4	.044	4.78
3.2	2.07	3.0	.038	4.81
3.3	2.14	2.6	.032	4.84
3.4	2.20	2.3	.027	4.86
3.5	2.27	2.0	.023	4.88
3.6	2.33	1.7	.020	4.90
3.7	2.40	1.5	.016	4.92
3.8	2.46	1.2	.014	4.93
3.9	2.53	1.1	.011	4.94
4.0	2.59	.89	.010	4.95
4.1	2.66	.75	.008	4.961
4.2	2.72	.63	.006	4.968
4.3	2.79	.53	.005	4.973
4.4	2.85	.44	.0043	4.978
4.5	2.92	.37	.0035	4.982
4.6	2.98	.31	.0029	4.986
4.7	3.05	.25	.0023	4.988
4.8	3.11	.21	.0019	4.991
4.9	3.18	.17	.0015	4.993
5.0*	3.2414	.136	.0012	5.0†

* Limit of standard range.
† 4.994 for the unlimited curve.

TABLE III

DEVIATIONS IN TERMS OF STANDARD DEVIATION AND *SR* DECILE DEVIATIONS

d/σ	*SR* decile deviation	d/σ	*SR* decile deviation	d/σ	*SR* decile deviation
.1	.15	1.1	1.70	2.1	3.24
.2	.31	1.2	1.85+	2.2	3.39
.3	.46	1.3	2.01	2.3	3.55
.4	.62	1.4	2.16	2.4	3.70
.5	.77	1.5	2.31	2.5	3.86
.6	.93	1.6	2.47	2.6	4.01
.7	1.08	1.7	2.62	2.7	4.16
.8	1.23	1.8	2.78	2.8	4.32
.9	1.39	1.9	2.93	2.9	4.47
1.0	1.54	2.0	3.08	3.0	4.63
				3.1	4.78
				3.2	4.94
				3.3*	5.09

* Beyond limit of standard range.

$1\sigma = 1.5425$ *SR* deciles. On the printed chart, $1\sigma = 15.4$ mm.

TABLE IV

FREQUENCY DECILE AND *SR* DECILE DEVIATIONS

Frequency decile deviation	*SR* decile deviation	Frequency decile deviation	*SR* decile deviation	Frequency decile deviation	*SR* decile deviation
.1	.04	1.6	.64	3.1	1.35
.2	.08	1.7	.68	3.2	1.41
.3	.12	1.8	.72	3.3	1.47
.4	.15	1.9	.76	3.4	1.53
.5	.20	2.0	.81	3.5	1.60
.6	.23	2.1	.86	3.6	1.67
.7	.27	2.2	.90	3.7	1.74
.8	.31	2.3	.95	3.8	1.81
.9	.35	2.4	.99	3.9	1.89
1.0	.39	2.5*	1.04	4.0	1.98
1.1	.43	2.6	1.09	4.1	2.07
1.2	.47	2.7	1.14	4.2	2.17
1.3	.51	2.8	1.19	4.3	2.28
1.4	.55	2.9	1.24	4.4	2.40
1.5	.59	3.0	1.30	4.5	2.54
				4.6	2.70
				4.7	2.90
				4.8	3.17
				4.9	3.58
				5.0	5.00†

* Frequency quartile.
† Infinity for the unlimited curve.

TABLE V

SAMPLING PROBABILITY AND *SR* DECILE DISTANCES

P	*SR* decile distance	*P*	*SR* decile distance
1.0	0	.04	3.17
.9	.19	.03	3.35
.8	.39	.02	3.59
.7	.59	.01	3.97
.6	.81		
.5	1.04	.008	4.09
		.006	4.24
.4	1.30	.004	4.44
.3	1.60	.002	4.77
.2	1.98	.001	5.08*
.1	2.54		
.09	2.62		
.08	2.70		
.07	2.79		
.06	2.90		
.05	3.02		

* Beyond standard range.

TABLE VI

PERCENTAGE FREQUENCIES IN VARIOUS EQUAL DIVISIONS OF THE STANDARD RANGE

Number of classes in standard Range	Serial order of class	Included % of total frequency
4 (quartiles)	1, 4	5.2
	2, 3	44.7
5 (quintiles)	1, 5	2.5
	2, 3	23.2
	4	48.4
10 (deciles)	1, 10	.4
	2, 9	2.1
	3, 8	7.1
	4, 7	16.1
	5, 6	24.2
20 (vigintiles)	1, 20	.1
	2, 19	.3
	3, 18	.7
	4, 17	1.4
	5, 16	2.7
	6, 15	4.4
	7, 14	6.8
	8, 13	9.3
	9, 12	11.5
	10, 11	12.7

REFERENCES

PEARL, RAYMOND
1940. Introduction to medical biometry and statistics. Third edition. W. B. Saunders Co., Philadelphia and London. (The abscissal decile technique is not in previous editions.)

PEARSON, EGON S.
1932. The percentage limits for the distribution of range in samples from a normal population. ($n \leqslant 100$.) Biometrika, XXIV, pp. 404–417.

SIMPSON, GEORGE GAYLORD
1941. Range as a zoological character. Amer. Jour. Sci., CCXXXIX, pp. 785–804.

SIMPSON, GEORGE GAYLORD, AND ROE, ANNE
1939. Quantitative zoology. Numerical concepts and methods in the study of recent and fossil animals. McGraw-Hill Book Co., New York and London.

TIPPETT, L. H. C.
1925. On the extreme individuals and the range of samples taken from a normal population. Biometrika, XVII, pp. 364–387.

WEISENBURG, THEODORE, ROE, ANNE, AND MCBRIDE, KATHARINE E.
1936. Adult intelligence. A psychological study of test performances. The Commonwealth Fund, New York.